Logicism Revisited*

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1 INTRODUCTION

I was led to revisit logicism by an historical riddle, so I will begin with that. Mathematics always played a key role in the philosophical battle between empiricists and rationalists or intellectualists. The empiricists always had trouble with mathematics: some (like Locke) said it consisted of 'trifling' or 'verbal' propositions, others (like Mill) said it consisted of empirical truths (Hume vacillated between these two as regards geometry). Neither account seemed plausible. The intellectualists, on the other hand, derived their chief comfort and inspiration from mathematics. Anyone who denied that *a priori* reasoning could issue in genuine knowledge was met with the triumphant question 'What about Euclid's geometry?'. Russell describes the situation well (in his [1897], p. 1):

> Geometry, throughout the 17th and 18th centuries, remained, in the war against empiricism, an impregnable fortress of the idealists. Those who held—as was generally held on the Continent—that certain knowledge, independent of experience, was possible about the real world, had only to point to Geometry: none but a madman, they said, would throw doubt on its validity, and none but a fool would deny its objective reference. The English Empiricists, in this matter, had, therefore, a somewhat difficult task; either they had to ignore the problem, or if, like Hume and Mill, they ventured on the assault, they were driven into the apparently paradoxical assertion that Geometry, at bottom, had no certainty of a different *kind* from that of Mechanics . . .

Now the great achievement of modern empiricism, we are often told, is to have removed this old objection to empiricism. Modern empiricists have shown, it is said, that Locke was basically right: despite appearances, mathematics does consist of 'trifling propositions', or more precisely, of

* An earlier version of this paper was read at a meeting of the British Society for the Philosophy of Science on 13 October 1975. I have benefited from comments and criticisms, made at that meeting and elsewhere, by Peter Clark, Max Cresswell, Donald Gillies, Moshe Machover, Graham Oddie, Sir Karl Popper, Robert Stoothoff, Pavel Tichý, and John Worrall.

tautologies. The truths of mathematics are all *logical truths*. And the *a priori* status of these truths is no threat to empiricism because they are as empty of content as 'Either snow is white or it isn't'. The desperate solution of some early empiricists has now been shown to be the correct one by the detailed reduction of mathematics to logic. Such is the philosophical importance of the *logicist programme*. No wonder our modern empiricists call themselves 'logical empiricists' to distinguish themselves from their forbears who could not take philosophical advantage of the reduction of mathematics to logic.

Yet something is wrong with this empiricist success story. It is well-known that the programme of reducing mathematics to logic could not be carried through, that it foundered on the logical paradoxes. And yet, decades later, the logical empiricists made the logicist thesis a cornerstone of their position. It looks like a prime example of Georg Cantor's Law of the Conservation of Error: a *thesis* continues to lead a healthy life long after the *programme* in which it was embodied has passed away. And yet logical empiricists did not ignore the difficulties which beset the logicist programme—indeed, they were better aware of them than most. This, then, is my historical riddle.

My solution to the riddle is that the logicist thesis which survived into logical empiricism is a very *different* thesis from the original one. I will call this new thesis *If-thenism*, to distinguish it from old-style logicism or logicism proper.[1] In this paper I will show how If-thenism rose from the ashes of old-style logicism, explain the difference between them, and ask whether If-thenism is an adequate philosophy of mathematics.

2 OLD-STYLE LOGICISM AND ITS BREAKDOWN

Old-style logicism was an incredibly bold thesis. Russell stated it as follows, in his 'Mathematics and the Metaphysicians' written in 1901 (Russell [1917], pp. 75–6):

> It is common to start any branch of mathematics—for instance, Geometry—with a certain number of primitive ideas, supposed incapable of definition, and a certain number of primitive propositions or axioms, supposed incapable of proof. Now the fact is that, though there are indefinables and indemonstrables in every branch of applied mathematics, there are none in pure mathematics except such as belong to general logic . . . All pure mathematics—Arithmetic, Analysis, and Geometry—is built up by combinations of the primitive ideas of logic, and its propositions are deduced from the general axioms of logic . . . And this is no longer a dream or an aspiration. On the contrary, over the greater and more difficult part of the domain of mathematics, it has been already accomplished; in the few remaining cases, there is no special difficulty, and it is now being

[1] The term was coined by Putnam who, as we will see, defends a version of the doctrine.

rapidly achieved. Philosophers have disputed for ages whether such deduction was possible; mathematicians have sat down and made the deduction. For the philosophers there is now nothing left but graceful acknowledgements.

We can divide the thesis so confidently expounded here into three parts:

(A) all so-called primitive notions of mathematics can be defined using only logical notions;

(B) all so-called primitive propositions of mathematics can be deduced from logical axioms;

(C) all theorems of mathematics can be deduced from its so-called primitive propositions (and hence, by virtue of (B), from logical axioms).

The logicist programme was the monumental effort of Frege, and then of Russell and Whitehead, to give a detailed demonstration of these three claims.

Of course, some of the groundwork for this ambitious programme had already been done. Analysis had been 'arithmetized', various algebraic theories had been axiomatised, the axiomatisations of geometrical systems had been improved, Cantor and Dedekind had developed set theory, and Dedekind and Peano had axiomatised arithmetic. But in none of this earlier works was the logic involved made fully explicit. The early logicists proposed to remedy this defect, and show that mathematical proofs could be formalised (hence their thesis (C)). And they also proposed to define Peano's primitive arithmetical notions in logical terms (thesis (A)), and to deduce Peano's arithmetical axioms from logical axioms (thesis (B)). The task remained a monumental one, despite the work of their predecessors.

As is well-known, the logicist programme, and thesis (B) in particular, foundered upon the logical paradoxes. It turned out that one of the necessary axioms of logic, far from being a trivial logical truth, was logically false. The axiom in question, the (unrestricted) Axiom of Set Abstraction, states that there exists, for any property we describe *via* an open formula, a set of things which possess the property.[1] From this Axiom we can easily derive Russell's Paradox.[2] Hence something was wrong with the proposed logicist foundation for mathematics, and it had to be revised. Frege responded by amending his Basic Law (V), his version of the

[1] Frege had expressed doubts about the 'self-evidence' of his version of this axiom: "A dispute can only arise, as far as I can see, with regard to my Basic Law concerning courses-of-values (V), which logicians perhaps have not yet expressly enunciated, and yet is what people have in mind, for example where they speak of the extensions of concepts. I hold that it is a law of pure logic. In any event the place is pointed out where the decision must be made" (Frege [1964], pp. 3–4).

[2] Russell's system is, of course, also subject to a version of his paradox which involves only predicates. This version of the paradox shows the untenability of unrestricted quantification over predicate variables, just as the set-theoretical version shows the untenability of unrestricted or naive set theory.

Axiom of Abstraction, to try to avoid Russell's paradox. But the amended law was also contradictory. Frege probably discovered this himself, and as a result was finally led to abandon the logicist programme.[1]

Russell's solution to the problem was his famous *Theory of Types*. The unrestricted Axiom of Abstraction was renounced, thus avoiding Russell's Paradox and (hopefully) any other paradoxes. Unfortunately Russell's new logic, as well as preventing the deduction of paradoxes, also prevented the deduction of mathematics. Russell therefore supplemented it with some additional axioms, the Axioms of Infinity, Choice, and Reducibility, and he and Whitehead proceeded to show that the whole of classical mathematics could be obtained from the Theory of Types together with these additional axioms (here, of course, I ignore Gödelian complications). Showing this was, of course, a great achievement and one which, as Russell might say, the philosophers can but gracefully acknowledge.

Zermelo's solution to the problem was structurally similar. He too renounced the unrestricted Axiom of Set Abstraction, and proposed less powerful axioms for set theory. The hope was that these new axioms would be powerful enough to yield mathematics, but not so powerful as to yield contradictions. Zermelo did not know whether his second hope had been fulfilled (and Gödel later showed that in a sense we cannot know this). But it was different with the first hope. It did turn out that the whole of classical mathematics could be reduced to Zermelo's set theory: all mathematical notions were defined in terms of logical notions *together with* 'ϵ', the single primitive notion of set theory; and all true propositions of classical mathematics were derived from logical axioms *together with the axioms of set theory* (ignoring Gödelian complications once more). Again, a great achievement which the philosophers can but gracefully acknowledge.

Philosophers might well ask, however, what has become of the major *philosophical* claim of the early logicists. Does the reduction of mathematics to set theory (or to a theory like Russell's) establish that mathematics is a branch of *logic*? Clearly this will depend upon whether we count set theory (or Russell's theory) a branch of logic. I now turn to this question.

3 IS SET THEORY A BRANCH OF LOGIC?

The question sounds dangerously verbal. One might merely stipulate that the term 'logic' is to cover set theory, and then pronounce the logicist thesis true. But this is to make logicism true by arbitrary stipulation, a method which (as Russell might remark) has all the advantages of theft over honest toil. If the assimilation of set theory to logic is to be more than

[1] On the failure of 'Frege's way out' see Quine [1955].

verbal, it must involve showing that (*a*) the primitive notion of set theory, 'ϵ', is a *logical* notion, and that (*b*) the axioms of set theory are *logical* truths. (Similar things would have to be shown for Russell's theory.) Can either of these things be shown?

It is hard even to discuss the first question, whether 'ϵ' is a logical notion, for the simple reason that we lack a convincing account of what it takes for a notion to be a logical one. Tradition has sanctified a few notions as logical ones: the usual connectives, the quantifiers, the 'is' of predication, and (though some dispute its inclusion) the 'is' of identity. These are the notions that figure essentially in the usual rules of inference—and fixed meanings are assigned to them by the usual semantical rules. But what is the *rationale* behind this traditional list? Why do logicians count 'is' logical and 'eats' non-logical?

Bolzano, that great pioneer of the foundations of logic, despaired of an answer and relativised his definitions of logical truth, logical consequence, *etc.*, to an arbitrary selection of terms to be counted logical. Now any true statement comes out logically true if we count all its terms logical and so are not allowed to vary the interpretation of any of them. And a hallowed syllogism such as Barbara will come out invalid if we count 'are' non-logical and interpret it to mean, say, 'eat'. Bolzano found this quite acceptable: he could see no way of establishing that "All men are mortal" is not *really* a logical truth, or that Barbara is *really* valid.[1] Tarski drew attention to the problem in 1935, and concluded "no objective grounds are known to me which permit us to draw a sharp boundary between the two groups of terms" (Tarski [1956], pp. 418–19). Popper tried hard to solve the problem in the 1940s (see Popper [1947]), but has recently admitted that his solution does not work (see Popper [1974], p. 1096). Kemeny defines the logical notions as those whose meaning is fixed by the customary semantical rules (Kemeny [1956], part 1, p. 17), but admits that the rules are drawn up with a specific list of logical notions in mind and hence cannot provide a *rationale* for that list.

In this situation the prospects of settling our first question in a non-arbitrary way do not seem bright. There are, however, two *arguments* on the question, one for classing 'ϵ' logical and one against. But neither of them is very conclusive.

The early logicists did not hesitate to class 'ϵ' logical. And we can reconstruct the following argument for doing so. Since the Axiom of Set Abstraction is our sole existential axiom for sets, each of our sets is determined by an open formula. So we can eliminate 'ϵ' wherever it occurs in favour of admittedly logical notions contained in the open formula. To

[1] On Bolzano see Kneale and Kneale [1962], pp. 365–71.

put it crudely, 'ϵ' is merely an alternative notation for admittedly logical notions, principally the 'is' of predication. Or as Frege put it (Frege [1972], p. 32):

I have replaced the expression 'class' [or, we might add, 'member of a class'] which is often used by mathematicians, by the expression 'concept' [or, we might add, 'falling under a concept'] which is customary in logic; and this is not merely an indifferent change of nomenclature, but is important for the knowledge of the true state of affairs.

The argument is cogent enough, but it rests on the mistaken assumption that the Axiom of Abstraction is true. The discovery of the paradoxes undermined this argument. And in axiomatic set theory 'ϵ' is counted a non-logical or primitive mathematical notion.

It remains the case, however, that in Russell's Theory of Types 'ϵ' is an explicitly defined notion (on each type level). If we count all notions in the *definiens*, and in particular the higher-order quantifiers, as logical, then presumably 'ϵ' is to be counted logical also. Carnap and Hempel do so, and triumphantly conclude that all mathematical notions are logical ones.[1] Quine demurs, arguing that the "tendency to see set theory as logic has depended early and late on overestimating the kinship between membership and predication". Predication is one thing, says Quine, but once we existentially quantify a predicate variable we assert the existence of an *attribute*, and *via* that attribute, of a set. He concludes that so-called higher-order predicate calculus is actually a "way of presenting set theory [which] gives it a deceptive resemblance to logic" (Quine [1970], pp. 66–8). Presumably, for Quine, logic stops at first-order logic, and the higher-order quantifiers and 'ϵ' are not to be counted logical notions. Is this a mere prejudice in favour of first-order logic? This brings me to the argument against classing 'ϵ' logical.

The argument rests on Gödel's results that first-order logic is complete while second-order logic is not. If we stick to the traditional list of logical notions, and define the notion of logical consequence accordingly, then all the logical consequences of a set of premises can be captured by syntactic methods. If, on the other hand, we extend the list of logical notions to include higher-order quantifiers, then the logical consequences of a set of premises can no longer be captured syntactically.[2] Therefore, the argument runs, we should refuse to extend the title 'logic' to so-called higher-order logics. The argument contains a tacit and unargued assump-

[1] See Carnap [1942], section 13, pp. 57–8; Hempel [1945], p. 375.

[2] For an informal account of these results, see Henkin [1967]. I here ignore the so-called 'completeness theorem' for higher-order logics: this arises from the attempt to give a semantic characterisation of the syntactically provable sentences, and involves a distortion of the notion of logical consequence.

tion: that logic must be confined to what can be captured syntactically. Hence it is not a conclusive argument.

So we see that it is not easy to settle the question of whether 'ϵ' is a logical notion. Perhaps we should accept the view, once expressed by Tarski, that this is a matter of taste: if you prefer to work in the Theory of Types you will count 'ϵ' logical, if you prefer to work in set theory you will not.[1] I do not think that it really matters if we reach this rather sad conclusion, because it is our second question, whether the axioms of set theory are logical truths, which is the really crucial one.

As will be apparent from this, I think our two questions are *independent* of each other, so that we could count 'ϵ' logical without counting the axioms of set theory logical truths. Many philosophers would disagree. Hempel claims that because the Axiom of Infinity "is capable of expression in purely logical terms [it] may be treated as an additional postulate of logic" (Hempel [1945], p. 377). Kemeny claims that most logicians recognise the Axioms of Infinity and Choice "as legitimate logical principles" presumably for the same reason (Kemeny [1959], p. 21). This is a position which goes back to Wittgenstein's *Tractatus*, and to Ramsey.

But there is a simple argument against it, which goes back to Russell. First, a logical truth may contain non-logical notions (consider "All men are men"), so that containing only logical notions is not a *necessary* condition for being a logical truth. Second, and more controversially, it is not a *sufficient* condition either: for we can express, using only admittedly logical notions, each of the mutually incompatible claims "There is exactly one thing", "There are exactly two things", "There are exactly three things", and so on; can it plausibly be maintained that one of these is logically true, and the rest logically false![2]

[1] Tarksi expressed this view in a lecture 'What are logical notions?' delivered in London on 16 May 1966. The basic idea of that lecture was that the logical notions are those which are invariant under every one-one transformation of the 'universe of discourse' onto itself (which goes back to a paper of Lindenbaum and Tarski of 1935: see Tarski [1956], chapter XIII). But this idea cannot settle the question of whether 'ϵ' is a logical notion.

[2] To say, for example, that there are exactly two things, we can write: $(\exists x)(\exists y)(x \neq y \; \& \; (z)(z = x \vee z = y))$.

Wittgenstein regarded any such proposition, Russell's Axiom of Infinity included, as a nonsensical pseudo-proposition which was trying to say what could only be shown: see his [1922], 4·1272 (also 2·022–2·023 and 5·534–5·535). Ramsey thought such propositions, including Russell's Axioms of Infinity and Choice (though not the Axiom of Reducibility), were either tautologous or contradictory, though the human mind may never be able to discern which: see his [1931], pp. 57–61.

The rationale of Ramsey's view appears to be this. The truth or falsehood of existential claims like this hinges on the cardinality of the domain of interpretation. By varying the cardinality of the domain, we can make any such statement come out true in one interpretation and false in another. But suppose we insist that part of the *definition of a language* is the domain over which the quantifiers are to range, so that all interpretations of any sentence *of that language* must have the same domain. Then any existential

Russell did not think so, and for a simple reason. Each of these statements makes a specific existential claim which is false in some possible worlds, whereas "Pure logic . . . aims at being true . . . in all possible worlds, not only in this higgledy-piggledy job-lot of a world in which chance has imprisoned us" (Russell [1919], p. 192). It was for precisely this reason that Russell refused to count his Axioms of Infinity, Choice, and Reducibility as logical truths. When he wrote the *Principles of Mathematics* Russell still hoped that the Axiom of Infinity might be proved from logic.[1] But he came to regard it "as an example of a proposition which, though it can be enunciated in logical terms, cannot be asserted by logic to be true".[2] In *Principia Mathematica* the Axioms of Infinity, Choice, and Reducibility were said not to be logically necessary propositions, but rather propositions which "can only be legitimately believed or disbelieved on empirical grounds".[3]

This verdict of Russell's is preserved when we transform his rather vague Leibnizian talk about 'truth in all possible worlds' into our more precise semantical definition of logical truth. That definition states, roughly, that a statement is logically true if it comes out true in all interpretations in all (non-empty) domains.[4] Now everyone agrees that Russell's problematic

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claim *of our language* will be either true in all interpretations, hence logically true, or false in all interpretations, hence logically false.

But this is to make the notion of logical truth *relative to language* in an extreme fashion (though if you operate mistakenly with only one language, as Wittgenstein did in the *Tractatus*, the relativity is not apparent). On this view there are infinitely many first-order languages (one whose quantifiers range over one-element domains, a second whose quantifiers range over two-element domains, and so on). And as well as the formulas which are logical truths in *all* of these, there is an infinite sequence of formulas each of which comes out logically true in exactly one language and logically false in all others. It seems to me that we should avoid definitions of 'language' (hence of 'interpretation' and of 'logical truth') which have such odd results.

[1] See Russell [1903], Introduction to the second edition, p. viii (Russell tells us on p. v that most of the book was written in 1900). On the inadequacy of proposed 'proofs' of the Axiom, see Russell [1919], chapter XIII.

[2] Russell [1919], pp. 202–3. The same applied to the Axioms of Choice and Reducibility (Russell [1919], pp. 117, 191), and indeed, to all 'existence theorems' (Russell [1903], Introduction to the second edition, p. viii). Russell even came to regard it as "a defect in logical purity" that his logical axioms implied the existence of at least one thing (Russell [1919], p. 203, footnote); for this anomaly, see the next footnote but one.

[3] Russell and Whitehead [1910–13], volume II, p. 183 (on the Axiom of Infinity); see also volume I, p. 62 (on the Axiom of Reducibility), and volume I, p. 504 (on the Axiom of Choice). The 'empirical' or 'inductive' grounds for believing these axioms included the fact that true mathematical statements could be derived from them. Originally mathematics was to be saved from scepticism by being derived from trivially true logic—now 'logic' is to be saved from scepticism by having trivially true mathematics derived from it (see Lakatos [1962], pp. 174–8.

[4] The restriction of interpretations to *non-empty* domains is the source of the minimal existential logical truth 'There is at least one thing', since it renders valid the argument from the logical truth '$(x)(Fx \vee -Fx)$' to '$(\exists x)(Fx \vee -Fx)$'. Removing this 'defect in logical purity' leads to the so-called 'free logics'.

axioms, or the axioms of set theory, are not logical truths in this sense of the term. For example, when we prove the *independence* of the usual axioms for set theory, we find for each of them an interpretation in a non-empty domain in which it comes out false while the rest come out true.[1] This is why these axioms are classed as 'proper' or 'mathematical' axioms, and not as logical ones.

It might be objected that this argument is much too swift, since we can also prove the independence of admittedly *logical* axioms. To do this we provide unintended 'interpretations' of them in which the intended meaning of the logical terms (given by the usual rules of interpretation) is changed. Now, it might be argued, the axioms of set theory *implicitly define* the intended meaning of 'ϵ'. Hence any 'interpretation' which falsifies one of those axioms must be one in which the intended meaning of 'ϵ' is changed. If we confine ourselves to intended interpretations, then the axioms of set theory will come out true in all interpretations and hence logically true.

But the argument is obviously circular. If the axioms of set theory implicitly define 'ϵ', then trivially any interpretation which falsifies one of those axioms must distort its meaning. In this way any (consistent) set of axioms could be deemed logically true. I see no reason to suppose that the *meaning* of 'ϵ' has been changed in an interpretation which falsifies, say, the Power-Set Axiom.[2] Moreover, Cohen's results show that there are *alternative*, equally consistent, set theories, one in which the Generalized Continuum Hypothesis is an axiom, and another in which its negation is an axiom.[3] We cannot, on pain of contradiction, deem *all* set-theoretical axioms logically true. And how could we defend the claim that one of these theories has *logically true* axioms, and the other a *logically false* one?

[1] Indeed, it is a theorem of Zermelo–Fraenkel set theory (ZF) that, if ZF is consistent then for any axiom A of ZF there is a set in which all the axioms are true except A.

[2] Each of the existential axioms of set theory can be falsified by assigning to the 'ϵ'-relation a proper subset of the 'intended extension' of that relation. It is different with the Axiom of Extensionality, which states that two sets are identical if they have the same members and which makes no existence claim. Any interpretation in a universe of sets which falsifies this axiom could be said to involve a change in the meaning of 'ϵ', since this axiom can be described as a partial implicit definition of 'ϵ'. (Another exception might be the Axiom of Foundation, which excludes, among other things, any set being a member of itself.) It is the *existential* axioms which are problematic: to say that these help to implicitly define 'ϵ' is to say that we do not really know what 'ϵ' *means* before we know which sets *exist*. If we applied this view to the universal quantifier, we would hold that any variation in the cardinality of the domain yields a non-standard interpretation which which changes the meaning of 'All'. And this would lead to the Wittgenstein–Ramsey view discussed in n. 2, p. 105, *above*.

[3] For a non-technical account which exploits the analogy with alternative geometries, see Cohen and Hersh [1967]. The same point applies to the various set theories obtained by adjoining 'Strong Axioms of Infinity' to the usual axioms.

The conclusion I draw from all this is that set theory (or Russell's type theory with the additional axioms) ought *not* to be counted a branch of logic. To extend the customary list of logical notions to include 'ϵ' smacks of arbitrary fiat. But more importantly, even if we do count 'ϵ' logical it is only by deforming the customary notion of logical truth that the axioms of set theory (or Russell's additional axioms) can be counted logical truths. The logicists did not achieve their declared aim. Their great, if unintended, achievement was the reduction of classical mathematics to set theory (or to Russell's theory), both fundamental *mathematical* theories. This conclusion is far from new; indeed, many will feel that I have been labouring the obvious. Mostowski, for example, writes (in his [1965], p. 7):

The logicism of Frege and Russell tries to reduce mathematics to logic. This seemed an excellent programme, but when it was put into effect, it turned out that there was simply no logic strong enough to encompass the whole of mathematics. Thus what remained from this programme is a reduction of mathematics to set theory. This can hardly be said to be a satisfactory solution to the problem of foundations of mathematics since among all mathematical theories it is just the theory of sets that requires clarification more than any other.

The Kneales agree, saying that once Russell had to postulate the Axiom of Infinity the logicist thesis was destroyed (Kneale and Kneale [1962], p. 699):

There is something profoundly unsatisfactory about the axiom of infinity. It cannot be described as a truth of logic in any reasonable use of that term and so the introduction of it as a primitive proposition amounts in effect to the abandonment of Frege's project of exhibiting arithmetic as a development of logic.

Quotations like this could be multiplied. Even the early logicists themselves seem to have reached the same verdict. At any rate, Frege gave up the attempt to base arithmetic upon logic, and tried instead to give classical mathematics a *geometrical* foundation.[1] Even Russell, in his pessimistic moments, confessed that it was he and not 'the philosophers' who had to admit defeat. Being Russell, he did it gracefully; reflecting on his eightieth birthday, he saw the main achievement of his intellectual life in the following terms (Russell [1969], p. 220):

I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. ... But as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. Having constructed an elephant upon which the

[1] By 1924 Frege had come to the conclusion that "the paradoxes of set theory have destroyed set theory". He continued: "The more I thought about it the more convinced I became that arithmetic and geometry grew from the same foundation, indeed from the geometrical one; so that the whole of mathematics is actually geometry". (These two remarks are quoted by Bynum in his Introduction to Frege [1972]; *cf.* pp. 53–4.)

mathematical world could rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure than the elephant, and after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that *I* could do in the way of making mathematical knowledge indubitable. Then came the First World War, and my thoughts became concentrated on human misery and folly.

This makes the historical riddle with which I began all the more puzzling. If the logicist programme was a dead-duck by about 1920, how could the positivists make the logicist thesis a cornerstone of their position in the 1920s?

4 LOGICISM RESCUED BY THE IF-THENIST MANOEUVRE

It was actually Russell who found a way to rescue logicism from defeat; and the key to it was provided by the problem of assimilating *geometry* to logic. Frege had actually excluded geometry from the logicist thesis, and had endorsed Kant's view of it:

I consider KANT did great service in drawing the distinction between synthetic and analytic judgements. In calling the truths of geometry synthetic and a priori, he revealed their true nature. (Frege [1884], section 89, pp. 101–2)

Russell, on the other hand, thought that the discovery of non-Euclidian geometries had undermined Kant's original position, and in his first major publication he tried to rescue it. In his *Foundations of Geometry* of 1897, Russell sought what was common to Euclidean and non-Euclidean systems, found it in the axioms of projective geometry, and took a Kantian view of them. As for the additional axioms which distinguished Euclidean from non-Euclidean systems, these were empirical statements (Russell [1897], Introduction, section 9). But after he had adopted the logicist thesis, Russell sought a way to bring geometry into the sphere of logic. And he found it in what I shall call the *If-thenist manoeuvre*: the *axioms* of the various geometries do not follow from logical axioms (how *could* they, for they are mutually inconsistent?), nor do geometrical *theorems*; but the *conditional statements linking axioms to theorems* do follow from logical axioms. Hence geometry, *viewed as a body of conditional statements*, is derivable from logic after all. As Russell himself put it (in the Introduction to the second edition of his [1903], p. vii):

It was clear that Euclidean systems alike must be included in pure mathematics, and must not be regarded as mutually inconsistent; we must, therefore, only assert that the axioms imply the propositions, not that the axioms are true and therefore that the propositions are true.

Russell argued that the discovery of non-Euclidean geometries forced us to distinguish *pure geometry*, a branch of pure mathematics whose assertions are all conditional, from *applied geometry*, a branch of empirical science. After describing the emergence of non-Euclidian geometry, he says (Russell [1903], p. 373):

Geometry has become . . . a branch of pure mathematics, that is to say, a subject in which the assertions are that such and such consequences follow from such and such premisses, not that entities such as the premisses describe actually exist. That is to say, if Euclid's axioms be called *A*, and *P* be any proposition implied by *A*, then, in the Geometry which preceded Lobatchewsky, *P* itself would be asserted, since *A* was asserted. But nowadays, the geometer would only assert that *A* implies *P*, leaving *A* and *P* themselves doubtful.

In this way the axioms of the various geometries cease to be problematic for the logicist, because they cease to be asserted as axioms at all (let alone asserted to be derivable from logical axioms):

The so-called axioms of geometry, for example, when Geometry is considered a branch of pure mathematics, are merely the protasis in the hypotheticals which constitute the science. They would be primitive propositions if, as in applied mathematics, they were themselves asserted; but so long as we only assert hypotheticals . . . in which the supposed axioms appear as protasis, there is no reason to assert the protasis, nor, consequently, to admit genuine axioms. (Russell [1903], p. 430)

Russell's If-thenist construal of geometry does, in fact, have a long history. Descartes, anxious to render mathematical truths immune from most sceptical attacks, hinted in the *First Meditation* that they are all conditional and hence do not assert existence:

. . . Arithmetic, Geometry and other science of that kind which only treat of things . . . without taking any great trouble to ascertain whether they are actually existent or not, contain some measure of certainty and an element of the indubitable. (Descartes [1911], volume I, p. 147)

Locke agreed:

All the discourses of the mathematicians about the squaring of the circle, conic sections, or any other part of mathematics, concern not the existence of any of these figures: but their demonstrations, which depend on their ideas, are the same, whether there be any square or circle existing in the world or no. (Locke [1690], book IV, chapter iv, section 8)

And Leibniz, that great forerunner of logicism, echoed the point:

As to *eternal truths*, it is to be noted that at bottom they are all conditional, and say in effect; Granted such a thing, such another thing is. For instance, when I say 'Every figure which has three sides will also have three angles', I say nothing but this, that supposing there is a figure with three sides, this same figure will have three angles. (Leibniz [1916], book IV, chapter 11, section 14)

Not only does Russell's position have a long history. It is also a position which is widely accepted *at least as regards geometry* by many who would not regard themselves as logicists or logical empiricists.

Now having already applied the If-thenist manoeuvre to the problematic axioms of geometry, it was natural for Russell to apply it also to the problematic Axioms of Reducibility, Infinity, and Choice.[1] These were not logical truths, since they made specific existence claims which were false in some 'possible worlds'. Hence postulating them *as axioms* would mean the end of the logicist programme. But conditional statements linking them to 'theorems' derivable from them will still be derivable from logic *provided that* the derivation of the 'theorem' from the 'axioms' was correct. This is exactly the course which Russell took. In 'Mathematical Logic as Based on the Theory of Types', written in 1908, he confessed that he could not prove the Axiom of Choice from logic and would therefore "state it as a hypothesis on every occasion on which it is used" (see Russell [1956], p. 99). The same manoeuvre occurs in *Principia Mathematica* regarding the problematic 'axioms'. Of the Axiom of Choice Russell and Whitehead say (Russell and Whitehead [1910–13], volume I, p. 504):

> We have not assumed its truth in the general [non-finite] case where it cannot be proved, but have included it in the hypotheses of all propositions which depend upon it.

And of the Axiom of Infinity they write (Russell and Whitehead [1910–13], volume II, p. 183):

> This assumption, like the multiplicative axiom [Axiom of Choice], will be adduced as a hypothesis wherever it is relevant. It seems plain that there is nothing in logic to necessitate its truth or falsehood, and that it can only be legitimately believed or disbelieved on empirical grounds.

Finally, Russell claimed that the If-thenist manoeuvre must be applied to any principle which is problematic from a logicist point of view:

> ... no principle of logic can assert 'existence' except under a hypothesis ... Propositions of this form, when they occur in logic, will have to occur as hypotheses or consequences of hypotheses, not as complete asserted propositions ... (Russell [1919), p. 204)

Clearly this would apply to all the problematic (existential) axioms of

[1] Even Frege had a brief flirtation with the idea. He amended his Basic Law (V) to avoid Russell's paradox. He then suggested that his amended law could be insulated from sceptical doubt if it were never *asserted* but rather always made the antecedent of conditional theorems, concluding "... even now I do not see how arithmetic can be scientifically founded, how numbers can be conceived as logical objects, unless we are allowed —at least conditionally—the transition from a concept to its extension" (Frege [1964], p. 127). Clearly, Frege never took the If-thenist manoeuvre too seriously—but Russell and the logical positivists were more enthusiastic about it.

set theory, so that set theory ***construed as a body of conditional statements*** might be shown to be derivable from logic as the logicist thesis requires.

By using the If-thenist manoeuvre, Russell arrives at a position which is far-removed from his original logicism. The claim that all so-called mathematical axioms can be deduced from logical axioms (thesis (B) of old-style logicism) is weakened to read:

(B*) *either* an apparently primitive proposition of mathematics can be deduced from logical axioms *or* it is not to be regarded as a primitive proposition at all but only as the antecedent of various conditional statements (all of which are derivable from logic in view of thesis (C)).

It turned out, in fact, that only *a fragment of arithmetic* (finite arithmetic) could be 'reduced to logic' in the way that old-style logicism demands.[1] The rest of mathematics could be 'reduced to logic' only if the If-thenist manoeuvre was applied to it first.

I think it fair to say that Russell never fully realised how far this new position was from logicism proper. And there was a special reason for this: his failure to distinguish a ***rule of inference***, a ***conditional statement*** of the form 'If A then B'. and a ***universal statement*** of the form '(x)(Fx ⊃ Gx)'. Some mathematical axioms fall into the third category. Applying the If-thenist manoeuvre results in statements of the second category. If we identify the two, we can still suppose that, even after adopting the If-thenist manoeuvre, we are deriving mathematical *axioms* from logic. And if we confuse both of these with rules of inference, we can suppose that we are deriving them from rules of inference. Because of these confusions, one always finds If-thenism rubbing shoulders with logicism proper in Russell's writings. As far back as 1901 we find a passage often smiled over but seldom understood, which immediately precedes the statement of logicism proper which I quoted on page 100 above:

Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true. Both these points would belong to applied mathematics. We start, in pure mathematics, from certain rules of inference, by which we infer that *if* one proposition is true, then so is some other proposition. These rules of inference constitute the major part of the principles of formal logic. We then take any hypothesis that seems amusing, and deduce its consequences. If our hypothesis is about *anything*, and not about some one or more particular things,

[1] As is admitted by Russell in the Introduction to the second edition of his [1903], p. viii. Not even Peano's axioms for arithmetic can be 'derived from logic' in the original logicist sense: that axiom which states that no two numbers have the same successor requires the 'Axiom of Infinity' for its proof (see Russell [1919], pp. 131–2).

then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. People who have been puzzled by the beginnings of mathematics will, I hope, find comfort in this definition, and will probably agree that it is accurate. (Russell [1917], p. 75)

It would be tedious to trace Russell's confusion through this passage, or through the similar passages at the outset of his *Principles of Mathematics* (see Russell [1903], chapter 1). Nor should we be too hard on Russell for them.[1] I mention them only because they often blinded him to the difference between his original thesis and his final one.

I now turn to the logical positivists, and to our historical riddle. The solution to the riddle is this: it was not old-style logicism which the positivists adopted, but rather logicism spiced with varying doses of If-thenism. Mind you, the *rhetoric* of old-style logicism persists, and is used as a stick to beat philosophical opponents. In 1930 Carnap begins by telling us that Whitehead and Russell had confirmed Frege's view that "mathematics is to be considered a branch of logic", in the following way:

It was shown that... every mathematical sentence (insofar as it is valid in every conceivable domain of any size) can be derived from the fundamental statements of logic... all the... sentences of arithmetic and analysis (to the extent that they are universally valid in the widest sense) are provable as sentences of logic. (Carnap [1930], pp. 140–1.)

The qualifications are, of course, crucial. The reader may wonder about all those mathematical sentences which are not "universally valid in the widest sense", which are not "provable as sentences of logic", and which make up the greater part of mathematics. Carnap says nothing to enlighten him about these. Instead, two pages later the unqualified thesis of old-style logicism is used as a stick with which to beat the opponents of empiricism:

Mathematics, as a branch of logic, is also tautological. In the Kantian terminology: The sentences of mathematics are analytic. They are not synthetic *a priori*. Apriorism is thereby deprived of its strongest argument. Empiricism, the view that there is no synthetic *a priori* knowledge, has always found the greatest difficulty in interpreting mathematics, a difficulty which Mill did not succeed in overcoming. This difficulty is removed by the fact that mathematical sentences are neither empirical nor synthetic *a priori* but analytic. (Carnap [1930], p. 143)

[1] Especially not when we reflect that the many upholders of the so-called 'inference-license' view of universal statements are victims of the same confusion. And when we reflect also that the very great difference between old-style logicism and If-thenism is frequently overlooked: for two examples among many see Pap [1949], pp. 108–9, or Robinson [1964], pp. 83 and 85.

Carnap is more forthcoming about the breakdown of the original logicist programme and how he proposes to deal with it a year later, in his paper 'The logicist foundations of mathematics'. Again, the opening rhetoric is that of old-style logicism: we are told, for example, that "The *theorems* of mathematics can be derived from logical axioms through purely logical deduction" (Carnap [1931], p. 31). But it soon transpires that 'theorem' does not mean what mathematicians standardly mean by it when they speak, for example, of the Prime Number Theorem or Pythagoras Theorem. For Carnap tells us of the discovery of the paradoxes, the theory of types, and the necessity to introduce the Axioms of Infinity and Choice. Then he continues:

> Russell was right in hesitating to present them as logical axioms, for logic . . . cannot make assertions about whether something does or does not exist. Russell found a way out of this difficulty. He reasoned that since mathematics was also a purely formal science, it too could make only conditional, not categorical, statements about existence: if certain structures exist, then there also exist certain other structures whose existence follows logically from the existence of the former. For this reason he transformed a mathematical sentence, say S, the proof of which required the axiom of infinity, I, or the axiom of choice, C, into a conditional sentence; hence S is taken to assert not S, but $I \supset S$ or $C \supset S$, respectively. This conditional sentence is then derivable from the axioms of logic. (Carnap [1931], pp. 34–5)

Here, then, Carnap takes Russell's way out of the dilemma. The problematic Axioms of Infinity and Choice (or of set theory, or of geometry) cease to be problematic because they cease to be *axioms* at all.[1]

Carnap changed his mind later, however, and decided that the Axioms of Infinity and Choice were analytic after all. He justified this in the case of the Axiom of Infinity by taking it to assert the existence, not of infinitely many *objects*, but of infinitely many *positions in space* (see Carnap [1937], pp. 141–2). It is unclear to me (and to Russell and Copi) why "the existence of infinitely many *positions* is less an empirical question than the existence of infinitely many *objects*" (Copi [1971], p. 67). And even if we accept Carnap's view of the Axiom of Infinity, the problematic Axiom of Choice remains. Yet in 1939 Carnap returns to the old-style logicist thesis that "all mathematical signs become logical signs, all mathematical theorems L-true propositions" (Carnap [1939], p. 48). This does not apply, however, to the theorems of the various geometries, to which the If-thenist manoeuvre is applied.[2]

This later position of Carnap's is also endorsed by Hempel in his paper

[1] The same position seems to have been adopted by Behmann in his [1934].

[2] See Carnap [1963], pp. 49–50; on pp. 47–8 Carnap reaffirms his view that the Axioms of Infinity and Choice are analytic.

'On the nature of mathematical truth' of 1945. As we already saw, Hempel classes 'ϵ' as a logical term (without argument), and concludes that the Axiom of Infinity "may be treated as an additional postulate of logic" on the (unargued) ground that "it is capable of expression in purely logical terms" (the same applies, it turns out, to the Axiom of Choice). Hempel then propounds:

> . . . *the thesis of logicism concerning the nature of mathematics*:
>
> Mathematics is a branch of logic. It can be derived from logic in the following sense:
>
> a. All the concepts of mathematics, i.e. of arithmetic, algebra, and analysis, can be defined in terms of four concepts of pure logic.
>
> b. All the theorems of mathematics can be deduced from those definitions by means of the principles of logic (including the axioms of infinity and choice). (Hempel [1945], pp. 377–8.

But Hempel is aware of the problems posed for old-style logicism by geometry and related fields—'mathematics' here does not include:

> . . . those mathematical disciplines which are not outgrowths of arithmetic and thus of logic; these include in particular topology, geometry, and the various branches of abstract algebra, such as the theories of groups, lattices, fields, etc. Each of these disciplines can be developed as a purely deductive system on the basis of a suitable set of postulates. If P be the conjunction of the postulates for a given theory, then the proof of a proposition T of that theory consists in deducing T from P by means of the principles of formal logic. What is established by the proof is therefore not the truth of T, but rather the fact that T is true provided that the postulates are. (Hempel [1945], p. 380)

Thus Hempel does not apply the If-thenist manoeuvre to the Axioms of Infinity and Choice—but he does apply it to the 'axioms' of large portions of mathematics.

There were, or course, other positivists who merely affirmed old-style logicism without mentioning its difficulties. Hahn, writing in 1933, contents himself with the declaration that, despite appearances, all mathematical propositions are tautologies, true by virtue of the meanings of the signs they contain. He seems barely aware of the difficulties logicism had encountered:

> To be sure, the proof of the tautological character of mathematics is not yet complete in all details. This is a difficult and arduous task; yet we have no doubt that the belief in the tautological character of mathematics is essentially correct. (Hahn [1933], p. 158)

Ayer's *Language, Truth and Logic*, first published in 1936, contains a similar account. Mathematical propositions are, he says, all analytic:

> . . . the criterion for an analytic proposition is that its validity should follow

simply from the definition of the terms contained in it, ... this condition is fulfilled by the propositions of pure mathematics. (Ayer [1936], p. 82)

Geometrical propositions are, however, treated in accordance with the If-thenist manoeuvre (see Ayer [1936], pp. 76, 82–4). Ayer does dissent from old-style logicism on one point:

A point which is not sufficiently brought out by Russell ... is that every logical proposition is valid in its own right. Its validity does not depend on its being incorporated in a system, and deduced from certain propositions which are taken as self-evident ... The fact that the validity of an analytic proposition in no way depends on its being deducible from other analytic propositions is our justification for disregarding the question whether the propositions of mathematics are reducible to propositions of formal logic, in the way that Russell supposed. For even if ... it is not possible to reduce mathematical notions to purely logical notions, it will still remain true that the propositions of mathematics are analytic propositions. They will form a special class of analytic propositions, containing special terms, but they will be none the less analytic for that. (Ayer [1936], pp. 81–2)

Here Ayer dissents from the thesis that all mathematical notions are definable from purely logical ones (thesis (A) of old-style logicism). But he does not, I think, dissent from the other two theses. His idea seems to be that mathematical statements are like "All men are men", which contains a term ('men') not definable in logical terms but which is still logically true; or like "All bachelors are unmarried", which also contains non-logical terms but which becomes a logical truth when we replace a defined term ('bachelor') by its *definiens* ('unmarried man'). The derivability of mathematics from logic is not *denied.* Ayer merely insists that a truth is not analytic *because* it is derivable from logic, but because of its logical form:

For it is possible to conceive of a symbolism in which every analytic proposition could be seen to be analytic in virtue of its form alone. (Ayer [1936], p. 81)

Like Hahn, then, Ayer simply ignores the breakdown of old-style logicism.[1]

[1] Donald Gillies takes a different view of the Hahn–Ayer position, in an unpublished paper of his called 'Logicism and the Logical Positivists' which was stimulated by an earlier version of the present paper and of which he kindly sent me a copy. Gillies thinks that the 'analytic view of mathematics', the view that mathematical propositions are true by virtue of the meanings of the words they contain, is a *different* view from logicism. He admits that the view is *vague,* and hopes to make it less so by developing a Wittgensteinian theory of meaning. I doubt that he will succeed. I think the only way to make 'analyticity' precise is to identify the analytic statements with statement which are either (*a*) logical truths, or (*b*) statements which become logical truths when conventionally defined terms are replaced by their defining terms. (It is perhaps worth adding that none of Quine's strictures against 'analyticity', in his famous [1951], apply to such a construal of it: Quine merely points out that what is 'analytic' in a *natural language* is vague, because what is conventionally defined in terms of what in natural language is vague. Quite so. But I have never quite understood how *this* shows that *logical truths,* as

I conclude that, in so far as the logical positivists had a defensible philosophy of mathematics *at all* (old-style logicism not being defensible), it was logicism spiced with varying amounts of If-thenism. Now this view of mathematics is a far less potent *philosophical* weapon than old-style logicism. The latter sought to show that, appearances notwithstanding, statements like "There are infinitely many primes" are logical truths, hence true in all possible worlds, hence factually empty. If this had succeeded, it would have been a very powerful argument indeed for the basic thesis of logical empiricism, the analytic/synthetic dichotomy. But it failed, and with the position the positivists actually adopted (their rhetoric aside) the situation is very different. The positivist confronts statements like "There are infinitely many primes", sees that they are neither synthetic nor analytic truths *as his basic thesis requires*, and therefore refuses to regard them as assertible statements at all. Using the If-thenist device, he construes all apparent assertions of statements which upset his central dogma as disguised conditionals, and claims that these are logically true. This is not an *argument for* the central dogma of positivism—it is a result of *applying it* to problematic cases.

So applying the If-thenist manoeuvre gives us a position which is of less philosophical interest than its predecessor. Some might be tempted to say that old-style logicism was a bold and exciting thesis which sadly turned out to be wrong, while its offspring is a puny imitation of it generated out of an *ad hoc* device to save the parent from defeat. But philosophers ought perhaps to love truth more than excitement. And they might well reply that the offspring, though less exciting than its parent, has the great advantage of being *true*. I discuss this question in my last section.

5 IF-THENISM

So far we have only considered the If-thenist manoeuvre in its historical setting. Thus considered, it does seem *ad hoc*, being applied piecemeal only to mathematical statements which turned out to be problematic for the old-style logicist. It is high time that we removed it from this historical setting, and let it stand on its own two feet. We then arrive at a more thoroughgoing position, which can be expressed by the following two claims:

(F) a mathematical statement is a conditional statement with a conjunction

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opposed to views about what is 'analytic' in some natural language, are open to revision in the light of empirical evidence.)

But whether or not Gillies succeeds in making his Wittgensteinian view of 'analyticity' precise, I doubt that it was the view of Hahn or Ayer. They simply ignored, or were ignorant of, the collapse of old-style logicism.

of 'mathematical axioms' as antecedent and a 'mathematical theorem' as consequent;

(G) all true mathematical statements can be deduced from logical axioms.

I will call this position *If-thenism.* And I will call (F) the *If-thenist prohibition*, since it prohibits the *pure* mathematician from asserting the truth of any of his 'axioms' or 'theorems'. (Anybody who asserts the truth of a mathematical 'axiom' or 'theorem' is, according to (F), an *applied* mathematician and his assertion is at bottom an *empirical* one.)

If-thenism is a much weaker position than old-style logicism. All that remains of old-style logicism is claim (C), which has become claim (G). The two are equivalent by virtue of the Deduction Theorem: a mathematical 'theorem' is deducible from logical axioms together with (closed) mathematical 'axioms' just in case the conditional statement linking the mathematical 'axioms' to the mathematical 'theorem' is deducible from logical axioms alone. Claim (C), or claim (G), is not a trivial claim: what it says is that all mathematical proofs can be formalised. The If-thenist will maintain that a real achievement of the early logicists was to have shown that claim (G) is correct. (A second real achievement was the unification of classical mathematics under set theory or the theory of types, which the If-thenist will regard as a primarily *mathematical* achievement.)

But an If-thenist will regard the other enterprises of the early logicists as misguided. The early logicists tried to establish their claims (A) and (B) for Peano's arithmetic. But an If-thenist does not *need* to try to define Peano's primitive notions in logical terms, nor does he *need* to try to derive Peano's axioms from logical axioms. All that the If-thenist needs to do in order to bring arithmetic into the sphere of logic is show that Peano's 'theorems' really can be formally derived from his 'axioms'. Similarly, all the logicists worries about the axioms of Infinity and Choice (or the axioms of set theory) are misguided from an If-thenist point of view. The mathematician merely derives 'theorems' from these 'axioms', he does not assert that the 'axioms' are true (let alone *logically* true).

So If-thenism is weaker than logicism proper—but is it true? At the heart of it is the If-thenist prohibition, (F), which says that the pure mathematician does not assert the truth of his 'axioms' or 'theorems', but only that of conditionals linking the two. Now I have not been around with a tape-recorder, but I suspect that many working mathematicians would not accept this prohibition, which turns them into rather sophisticated logicians. However, sociological facts about mathematicians are not philosophical arguments. An If-thenist need not be too impressed, even if an exhaustive survey of mathematicians should reveal that not one of them accepts his philosophy. He might reply that, just as fish are good swimmers but are not much good at hydrodynamics, so also good mathematicians are not much good at the philosophy of mathematics.

Mathematicians might, however, have a good reason for rejecting the If-thenist view of mathematics. For it applies, straightforwardly, only to *axiomatised* portions of mathematics. But mathematicians do creative work in areas which have not yet been axiomatised: think of geometry before Euclid (or perhaps Hilbert), analysis before Cauchy (or perhaps Weierstrass), or arithmetic before Peano (or perhaps Frege). If-thenism has nothing to say about un-axiomatised or pre-axiomatised mathematics, in which many creative mathematicians work. Therefore, *even if* its account of axiomatised mathematics is acceptable, as an account of mathematics as a whole it is seriously defective.

One philosopher of mathematics would have accepted this argument, and taken it even further. Imre Lakatos, in the Introduction to his *Proofs and Refutations*, attacks what he calls 'formalism', the identification of mathematics with formally axiomatised systems (and of the philosophy of mathematics with meta-mathematics or the study of such systems). According to Lakatos, formalism excludes from consideration all creative, growing mathematics. A mathematical theory can be formally axiomatised only *after* the creative mathematical work is done and the theory has ceased to grow: formal axiomatisation is, one might say, the kiss of death which turns a living thing into a museum piece. Lakatos does not deny that creative mathematics can be done *about* formally axiomatised theories by meta-mathematicians: but the formally axiomatised theory is the subject-matter, and the work is done in an informal meta-mathematical theory:

> Nobody will doubt that some problems about a mathematical theory can only be approached after it has been formalized, just as some problems about human beings (say concerning their anatomy) can only be approached after their death. But few will infer from this that human beings are 'suitable for scientific investigation' only when they are 'presented in "dead" form', and that biological investigations are confined in consequence to the discussion of dead human beings—although, I should not be surprised if some enthusiastic pupil of Vesalius in those glorious days of early anatomy, when the powerful new method of dissection emerged, had identified biology with the analysis of dead bodies. (Lakatos [1976], p. 3, note 3)

Now since If-thenism applies straightforwardly only to axiomatised theories, Lakatos would presumably regard it, not as a philosophy of mathematics, but as a philosophy of dead mathematics. We saw how If-thenism grew historically out of the basic dogma of logical empiricism. Lakatos claims that the exclusion of informal mathematics from mathematics stems from the same source:

'Formalism' is a bulwark of logical positivist philosophy. According to logical positivism, a statement is meaningful only if it is either 'tautological' or empirical. Since informal mathematics is neither 'tautological' nor empirical, it must be meaningless, sheer nonsense. The dogmas of logical positivism have been detrimental to the *history and philosophy of mathematics*. (Lakatos [1976], pp. 2–3)

But does the existence of informal, pre-axiomatised mathematics present an insuperable obstacle to the If-thenist view? To answer this question, let us look briefly at Lakatos's own account of informal mathematics, as presented in *Proofs and Refutations*.

Lakatos describes how, often for quasi-empirical reasons, mathematicians get interested in certain mathematical entities: the geometer in plane figures or polyhedra, the arithmetician in prime numbers, the analyst in areas under curves. They propose *conjectures* about these entities, and try to *prove* them. Both the conjectures and the proofs are criticised and, through intricate dialectical processes, improved. This process of trial and error (proof and refutation) results in a growing body of knowledge about the entities in question, organised in a more or less ramshackle deductive structure. An axiomatiser may then come along and look for a small number of true statements (axioms) from which all the other known truths in the field (theorems) can be derived.

What will an If-thenist say to an account such as this? He might well applaud it as a contribution to the *history* of informal mathematics, while insisting that it is *philosophically* question-begging. It speaks of the mathematician conjecturing, and trying to prove, that some categorical mathematical statement is true. Thus, for example, an arithmetician might conjecture, and try to prove, that there are infinitely many prime numbers. But the If-thenist denies that sense can be made of categorical claims like this, unless, of course, they amount to some sort of *empirical* claim. For him, to say that this proposition is true of the natural numbers is simply to say that it is deducible from the axioms which characterise the natural number sequence. Informal mathematics, so brilliantly depicted by Lakatos, is simply the process of *creating axiomatic structures*. And Lakatos's 'mathematical conjectures' are, at bottom, *logical* conjectures: to use Lakatos's own example, the Descartes–Euler conjecture that for all polyhedra $V-E+F = 2$ is simply the conjecture that the concept of 'polyhedron' can be defined in such a way that this proposition will be deducible from geometrical axioms together with the definition. Mathematical assertions, even the assertions of informal mathematics, are all disguised conditionals. What else, given that they are not *empirical* claims, could they be?[1]

[1] Having said this, the If-thenist might go on to disassociate himself from some of the excesses of 'formalism' to which Lakatos rightly objects. *Of course* (he might say)

The If-thenist challenge implicit in this question is not an easy one to answer. As an indication of this, I will consider Quine's quick dismissal of If-thenism. Using 'Hunt (sphere, includes)' to abbreviate Huntingdon's geometrical axioms, and 'T (sphere, includes)' to abbreviate a theorem deducible from those axioms, Quine writes:

But if as a truth of mathematics 'T (sphere, includes)' is short for 'If Hunt (sphere, includes) then T (sphere, includes)' still there remains, as part of this expanded statement, 'T (sphere, includes)'; this remains as a presumably true statement within some body of doctrine, say for the moment 'non-mathematical geometry', even if the title of mathematical truth be restricted to the entire hypothetical statement in question. The body of all such hypothetical statements describable as 'theory of deduction of non-mathematical geometry' is of course a part of logic; but the same is true of any 'theory of deduction of sociology', 'theory of deduction of Greek mythology', etc., which we might construct in parallel fashion with the aid of postulates suited to sociology or Greek mythology. The point of view toward geometry which is under consideration thus reduces merely to an exclusion of geometry from mathematics, a relegation of geometry to the status of sociology or Greek mythology; the labelling of the 'theory of deduction of non-mathematical geometry' as 'mathematical geometry' is a verbal *tour de force* which is equally applicable in the case of sociology or Greek mythology. To incorporate mathematics into logic by regarding all recalcitrant mathematical truths as elliptical hypothetical statements is thus in effect merely to restrict the term 'mathematics' to exclude those recalcitrant branches. But we are not interested in renaming. Those disciplines, geometry and the rest, which have traditionally been grouped under mathematics are the objects of the present discussion, and it is with the doctrine that mathematics in this sense is logic that we are here concerned. (Quine [1936], p. 327)

Now Quine rightly points to the *ad hoc* character of If-thenism: as he puts it, it is a "verbal *tour de force*" by which "recalcitrant mathematical truths" are regarded as "elliptical hypothetical statements". Quine is also quite right to point out that this "verbal *tour de force*" is equally applicable to sociology, Greek mythology, or any other empirical theory, since empirical theories too can be axiomatised. Of course, no If-thenist *does* apply the If-thenist manoeuvre in such cases: even the most ambitious logicist baulks at assimilating sociology or economics or physics or Greek mythology to logic. Hence he must have an *independent* reason for treating mathematics differently. And the logical empiricists did, of course, have such

informal mathematics is creative (though axiomatising a portion of mathematics is creative too, as is discovering and proving a new theorem in an axiomatic system). *Of course* (he might continue) the stratagems of informal mathematics are an important and fascinating field, as Lakatos has himself shown. *Of course* (he might add) there is a great difference between a more or less informal axiomatic system and a fully formalised one, and we are far from demanding that mathematics only be conducted in the latter. *But* (he might conclude) none of these concessions alters my central thesis: that mathematical assertions, properly construed, are all *conditional* in form and, if true, *logically* true.

a reason: their central dogma that there are empirical assertions and logical assertions, but nothing else. Theories of sociology or economics or physics or Greek mythology (even if axiomatised) fall into the first category: hence they are not given the If-thenist treatment.[1] But theories of mathematics (*pure* mathematics, not applied mathematics) do not: hence they are given the If-thenist treatment. What does the philosophical work here is the logical/empirical dichotomy, not If-thenism itself. This is simply the philosophical weakness of If-thenism, to which I already drew attention.

More to the point now, however, is Quine's own view of the "recalcitrant mathematical truths". He says that 'T (sphere, includes)' "remains as a presumably true statement within some body of doctrine". But this, it seems to me, is to fall back into If-thenism without noticing it. After all, a truth of sociology is simply *true*: it is not "true *within* some body of doctrine". What on earth does "true within some body of geometrical doctrine" *mean* if not "deducible from axioms characterising that body of doctrine"? Moreover, it is especially unfortunate that Quine chose a *geometrical* example. For there are *alternative* geometries, and Quine's 'T (sphere, includes)' could well be "true within Euclidean geometry" and "false within non-Euclidean geometry" (which is to say, of course, that it is deducible from Euclidean axioms, while its negation is deducible from non-Euclidean axioms). Anyone who says it is true *simpliciter* must, it seems, be making an *empirical* claim.

Moreover, what holds for geometry also holds for large portions of modern mathematics, which are concerned with investigating various axiom systems. In abstract algebra, topology, and set theory itself, mathematical results *do* seem to be conditional in form. Of course, this may not always be apparent from their formulation: "Division is unique in any field" (that is, "If F is a field, then division is unique in F"); "Complements are unique in distributive lattices" (that is, "If L is a distributive lattice, then complements are unique in L"); and so on. In all these cases we may claim that some group of axioms is true of some empirical subject-matter (after having given an empirical interpretation to the non-logical terms). But assertions like "Physical space is non-Euclidean" or "The

[1] I am not so sure about Greek mythology. For the ancient Greeks it was, presumably, a *factual* theory, and at least some of it was accepted as *true*. But for us (who *call* it 'mythology' and not 'theology') it is factually false. Yet we still want to say such things as "It is a truth of Greek mythology that Apollo was the son of Zeus and Leto". And the only way to make sense of such claims is to adopt an If-thenist construal of them (something like "If this-and-this basic assertion of the Greek myths holds, then Apollo was the son of Zeus and Leto"). Or do we want to say that Greek gods really do exist somewhere (though not, of course, in the real world), and that this assertion is true of them. Thus I think that we *do* construe Greek mythology in an If-thenist fashion, though the ancient Greeks presumably did not. (I think the problem of 'truth in fiction' or 'fictional truth' may be soluble along If-thenist lines—but that is another story.)

quantum-mechanical observables form a non-distributive lattice" belong, not to pure mathematics, but to empirical science. If-thenism challenges us to explain what we mean by saying that a mathematical axiom or theorem is true when this does *not* mean that it is an empirical truth.

I can think of one way to meet this challenge (though it is not one which would appeal to Quine). And that is to postulate, alongside the empirical realm, a realm of mathematical entities. A mathematical 'axiom' or 'theorem' can be true *simpliciter* because there is a realm of entities for it to be *true of*, quite independently of its deducibility or otherwise from other mathematical statements. (The resistance of mathematicians to If-thenism, if it exists, might well stem from their tacit adoption of a view like this one. And Lakatos's account of informal mathematics also seems to rely on some such view.)

But this kind of *naive* Platonism has many problems, not least of which is that posed by the existence of *alternative* mathematical theories. Are we to claim that actual 'mathematical space' (not *physical* space, but the space in our Platonic realm) is really Euclidean (or non-Euclidean), so that "The angle sum of a triangle is 180°" is really an unconditional mathematical truth (or falsehood)? Are we to claim that all lattices are (are not) distributive? Or that the 'universe of sets' in our Platonic realm does (does not) bear out the Continuum Hypothesis? I cannot make much sense of such claims. Nor, I suspect, could most mathematicians who would regard the investigation of Euclidean *and* non-Euclidean geometry, distributive *and* non-distributive lattices, Cantorian *and* non-Cantorian set theory, as all being equally legitimate from a *mathematical* point of view. Yet without such *naive* Platonistic claims we seem to be driven back to the If-thenist position.[1]

[1] I think, for example, that the sophisticated *evolutionary* Platonism of Popper need not trouble an If-thenist. Popper tries to combine a Platonistic view of the *objectivity* of human knowledge with the Darwinian view that human knowledge is an *evolutionary product*. Thus he insists that the natural numbers are a human creation (part and parcel of the creation of descriptive languages with devices for counting things), but that once created they become *autonomous* so that objective discoveries can be made about them and their properties are not at the mercy of human whim (see Popper [1972], pp. 158–61). An If-thenist could agree with much of this. We create, first of all, languages in which to express certain *empirical* claims: "Two apples placed in the same bowl as two other apples give you four apples"; "Two drops of water placed together give you one bigger drop of water"; *etc.* Then we come to treat numbers and their addition in a more abstract way (so that the second statement just given does not count as an empirical refutation of "$1+1 = 2$"). This is, at bottom, to create a more or less explicit collection of 'axioms' for the natural number sequence. And then we find that, once these are granted, we must also grant other statements about numbers like "There are infinitely many prime numbers". We *discover*, in other words, that our axioms have certain unintended logical consequences. The objectivity of mathematics is guaranteed by the fact that *what follows from what* is an objective question, and we need not postulate a realm of 'abstract mathematical entities' to ensure it.

However, we have yet to come to grips with some results which, it might be maintained, not only drive a further nail into the coffin of old-style logicism but also destroy its offspring If-thenism: *Gödel's incompleteness theorems*. Gödel showed that for any consistent (recursive) axiomatisation of arithmetic there is an arithmetical statement such that neither it nor its negation is *formally provable* from the axioms. And he showed that for any consistent (recursive) axiomatisation of arithmetic the statement that it is consistent cannot be formally proved within the system but must rely on methods stronger than those of arithmetic itself. How do these results bear upon old-style logicism and If-thenism?

There is no doubt that the early logicists thought that all arithmetical truths might be formally provable from arithmetical axioms (which were in turn to be formally proved from logical axioms). There is no doubt, in other words, that the phrase 'deduced from' in theses (B) and (C) of early logicism meant 'formally proved from', and that the early logicists identified arithmetical truth with provability from arithmetical axioms. Hence Gödel's first incompleteness theorem destroys claim (C) of early logicism just as effectively as the discovery of the paradoxes destroyed claim (B).

Neither is there any doubt, it seems to me, that the If-thenists carried over this logicist claim about formal provability into their thesis (G). They claimed, in other words, that all arithmetical truths could be formally proved from arithmetical axioms, so that the conditionals linking the two could be formally proved from logical axioms alone. Hence If-thenism, while it was not refuted by the earlier discovery of the paradoxes, seems to have been refuted at about the time it was proposed by Gödel's first incompleteness theorem. Has the If-thenist any effective reply to this criticism?

He might first try to stick to his guns, bringing Gödel's *completeness theorem for first-order logic* to his aid. This theorem shows that everything that follows from or is a logical consequence of any *first-order* axiomatic system can be formally proved from those axioms. If we identify logic with first-order logic, and mathematics with the collection of first-order theories, then we can continue to maintain the If-thenist position. A mathematical statement becomes, *via* claim (F), a conditional statement with a conjunction of first-order mathematical axioms as antecedent and a first-order mathematical theorem as consequent. And all the logically true conditionals of this sort will be formally provable from logical axioms alone. Mathematics, *in so far as it is adequately formalised in first-order logic*, can be identified with logic after all.

It will be objected, however, that Gödel's incompleteness theorems show precisely that first-order logic is *not* adequate for arithmetic (or for any mathematical theory strong enough to contain arithmetic). For consider the conditional statements 'P ⊃ G' and 'P ⊃ —G', where 'P' stands for the first-order axioms of arithmetic and 'G' for an undecidable Gödelian sentence. Or consider the conditional statement 'P ⊃ C', where 'C' is the arithmetical statement expressing the consistency of first-order arithmetic. None of these statements is first-order provable, and none of them is a first-order logical truth. Hence there are interpretations of the first-order axioms in which they are all true and G false (as well as interpretations in which all axioms are true and G true). And hence there are non-standard models of first-order arithmetic in which an 'arithmetical truth' (namely G) is false, so that these axioms are inadequate. Mathematicians think that the conditional statements 'P ⊃ G' and 'P ⊃ C' are *true*, because they think that their consequents are true. Hence there are conditional mathematical assertions which are not first-order logical truths. Thus 'first-order If-thenism' (as we might call it) collapses.

A second possible way out for the If-thenist is to renounce the claim that all the true conditionals of mathematics are formally provable, while continuing to maintain that they are *logical* truths.[1] This is to renounce the claim that first-order logic exhausts logic (since by the completeness theorem all first-order logical truths *are* provable). It enables the If-thenist to continue to maintain that all mathematical statements are conditional in form, and that the true ones are *logically* true. In particular, the conditional statements 'P* ⊃ G' and 'P* ⊃ C' (where 'P*' denotes the *second-order* axioms for arithmetic) continue to be logical truths. All that Gödel's incompleteness theorem shows, on this view, is that they are not logical truths which are provable from the axioms of second-order logic. It shows, in effect, that there *are* no axioms for second-order logic from which all the logical truths of second-order logic are provable.

This 'post-Gödelian If-thenism' (as we might call it) is a far cry from the original, whose central thesis was precisely the formal provability of all conditional mathematical truths. The post-Gödelian If-thenist could

[1] If I understand him rightly, this is the position Putnam defends in his [1967*b*] and his [1971]. In the former Putnam contrasts two "equivalent descriptions" of mathematics, the first the familiar "Mathematics as Set Theory", the second the unfamiliar "Mathematics as Modal Logic". Concerning the latter, Putnam claims that the mathematical content of a proof that Fermat's last theorem is false is expressible by a conditional scheme of modal logic of the form "Necessarily (A ⊃ —F)". But the reference to modal logic here seems to be a red-herring, since Putnam says that he is using necessity as being equivalent to logical validity (Putnam [1967*b*], pp. 9–11). I should also mention here Putnam's earlier [1967*a*], in which he coined the term 'If-thenism'. Incidentally, Putnam claims there that Russell subscribed to 'If-thenism' *before* he subscribed to logicism proper; as I see it, Russell never clearly distinguished between the two at all (see Putnam [1967*a*], p. 281).

still insist, however, that any *concrete mathematical result* is not only a logically true conditional, but a *provable* one. In particular, although Gödel showed that the consistency of arithmetic is not provable by methods formalisable within arithmetic itself (so that 'P* ⊃ C' is not a *provable* logical truth), Gentzen *has* proved the consistency of arithmetic from assumptions which transcend arithmetic. This result (like all others) is a provable logical truth 'P** ⊃ C', where 'P**' denotes the assumptions of Gentzen's proof. An objection to this is that mathematicians believe C to be true unconditionally, and not because it is provable from P** (which contains assumptions more dubious than C itself).[1] But *any* correct proof proceeds from assumptions which *ought* to be more dubious than the conclusion, since the conclusion is contained in them. The point of view which underlies this objection would, therefore, if generalised, undermine the whole notion of mathematical proof, and the axiomatic method with it. An If-thenist might well retort that he is interested in the status of mathematical results, and not in the strengths of the beliefs of mathematicians. And he might well reaffirm his conviction that all mathematical truths are logically true conditional statements.

This version of If-thenism stands or falls, it seems to me, on whether we are prepared to extend the title 'logic' to higher-order logic, and to countenance thereby logically true conditionals which are not provable. If we are, then If-thenism provides a way to assimilate mathematical truth to logical truth, without making the implausible claim that the various existential axioms of set theory or the various geometrical axioms are logical truths. But if we are not, if we are persuaded, for example, by the argument given on page 104, then we will have to admit that the question of the epistemological status of mathematics remains open.

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[1] This objection was raised by Moshe Machover, and endorsed by Donald Gillies in the paper referred to in n. 1, p. 116, *above* (assuming that I have understood them correctly).

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